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MATHEMATICAL PROBLEMS RAISED BY THE FLOOD DISASTER 1953.

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Mr Chairman, Ladies and Gentlemen,

Human fate and human will are not computable. Still, mathematics can soften nature's impact on human fate, and strengthen the effect of human response.

In the small hours of February 1st 1953 the South Western part of the Netherlands was stricken by a flood disaster unsurpassed in the memory of this country. Big floods, some of which causing heavy losses had occurred before, e.g. in 1916, when lack of manpower in wartime had partly been responsible for neglect of dikes. But we have at least to go back to the floods of 1825 and 1775, and perhaps even to those of 1570 and 1421 to find anything comparable with last year's one (for details see "De Ingenieur" [1953]).

According to data provided by Ir A. G. Maris over 150000 hectares of land were flooded. It caused a loss of over 1800 human lives; about 9000 buildings were demolished and 38000 damaged; there were 67 breaks of dikes, and hundreds of kilometers of dikes were heavily damaged. The total economic loss is estimated at 1,5 till 2 milliards of guilders. Also to a lesser extent parts of Great Britain and Belgium were galestricken and suffered considerable losses.

On the other hand it gave rise to perhaps the finest example of spontaneous national and international helpfulness hitherto seen in history. Help, in the form of voluntary labour and material goods, flowed in, not only from every part of our country, but from a large number of foreign countries also, and to such an enormous extent that we can not think of this proof of real unity of mankind without the deepest emotion.

In order to design measures for preventing similar disasters in future the government appointed a committee, consisting of prominent engineers with Ir A. G. Maris as Chairman. It is called the Delta-committee, because its realm is the whole delta formed by the rivers Rhine, Meuse and Scheldt. The renowned hydrogolist J. Th. Thijsse was immediately recalled from a lecturing tour in the U.S.A. The Delta-committee decided to base its proposals on a broad scientific basis, and therefore took several scientific institutions as advisers, like the Central Planning Bureau, the Royal Dutch Meteorological Institute, the Hydrological Laboratory of the Technical University at Delft, and the Mathematical Centre, and of course, several divisions of the Public Works Department itself. Between these institutions a fertile cooperation was established, in the form of working- and discussion groups, exchange of problems and results, etc.

Since then the breaks in the dikes have all been closed, already before the winter fell, the land has been reclaimed and drained, and an energetic beginning has been made to repair the other material damage. The Delta-committee has advised to close completely four of the six sea-arms in the South. Along the two other ones which lead to the harbours of Rotterdam and Antwerp, the dikes will be heightened, whereas a number of other works will be carried out.

The mathematical problems raised by the flood fall into three sets, belonging to different branches of mathematics, viz.

1. Statistical extrapolation problems concerning the distribution of the height of the sea-level;
2. Econometric decision problems concerning the height to which dikes should be heightened;
3. Problems in applied mathematics concerning the question which heightening of the sea-level is caused by a given depression moving over the North Sea.

The problems belonging to the first two groups are trivial from the mathematical point of view. Their interest for the mathematician lies solely in the logical analysis of the applicability of special mathemathical models. Those belonging to the third group, on the contrary, lead to rather difficult questions on partial differential equations, but are, as yet, still far from completely solved. Lack of time prevents me to go into the econometric problems, which I have investigated with the help of J. Kriens, and discussed extensively last month before the European Meeting of the Econometric Society at Upsala (D. van Dantzig [1954*b*]).

I shall give a short survey of the most important results hitherto obtained in the institutions mentioned before, giving somewhat more details about those, obtained in the Mathematical Centre, viz. in its Statistics Department in cooperative work under Hemelrijk and myself and in the Applied Mathematics Department under my own guidance.

The question might be asked why I venture to speak before an International Mathematical Congress about problems which partly are so elementary as to be almost trivial from a purely mathematical point of view, and, inasfar as they are mathematically interesting, have hitherto led to partial results only, most of which, moreover, are not my own.

Big floods, even far bigger and more disastrous ones, have occurred in many other countries also, even quite recently, and an event which has shaken our country to its roots may look small on a world wide scale. We, who have so recently suffered from such an event, are first to sympathize with the victims in other parts of the world. So we rather would not miss any possibly existing chance that results obtained and methods used here for solving the mathematical problems involved might be of some use in other similar cases also.

Although it might be considered as a national honorary duty of the Netherlands to further the solution of this problem, it is, as we shall see, so big that with restricted intellectual manpower available in a small country like ours during the time we are working on it (one year) only some modest results could be obtained. Because of the urgency of flood prevention in so large parts of Europe, Asia and America, this could be regarded as a typical instance of a problem where some form of international cooperation would be useful. Uniting internationally intellectual capacity and pooling mathematical knowledge would doubtless shorten considerably the time needed for solving the underlying problems.

This is not a matter of large scale computing only. In **1932** (D. van Dantzig [1932]) I have expressed the hope that there is one difference, at least, between mathematicians and horses. Horses have been abolished and replaced by cars and tractors, but mathematicians, I said, need not be afraid of being abolished in order to be replaced by computation machines. Although since then the area of mechanization of mathematics has set in, I still think this to be true, and the hydrodynamical problems needed for flood prevention offer a typical instance where the job of a mathematician does not consist of just "feeding a machine", and which, although it ultimately certainly will require large scale computing, in the present phase rather has need of what might be called "large scale mathematics".

These combined national and international aspects of the problem may, I hope, be considered as a justification of my speaking before this congress.

Statistical problems.

Formerly, engineers based the dike building on the highest flood hitherto observed. In the thirties, however, engineers became aware of the weakness of this method. In **1939** the government appointed a committee to investigate the question, which measures had to be taken in order to increase the security offered by our dike-system. Had not the german invasion and its aftermath, the reconstruction period needed after the war, prevented to carry out the measures it proposed, the 1953 flood would have caused no disaster. It struck us unprepared because it came too soon after the war.

In his by now renowned paper of 1939 P. J. Wemelsfelder [1939] determined a statistical estimate of the (cumulative) distribution of the sea-level heights during high tide at Hoek van Holland. Based on the observations during the period 1888–1937 he found that the exceedance frequencies $n(h)/n$, where $n(h)$ is the number of exceedances of the level h during n years, when plotted on logarithmic paper, very closely followed a straight line (fig. 1) and drew important conclusions from this result.

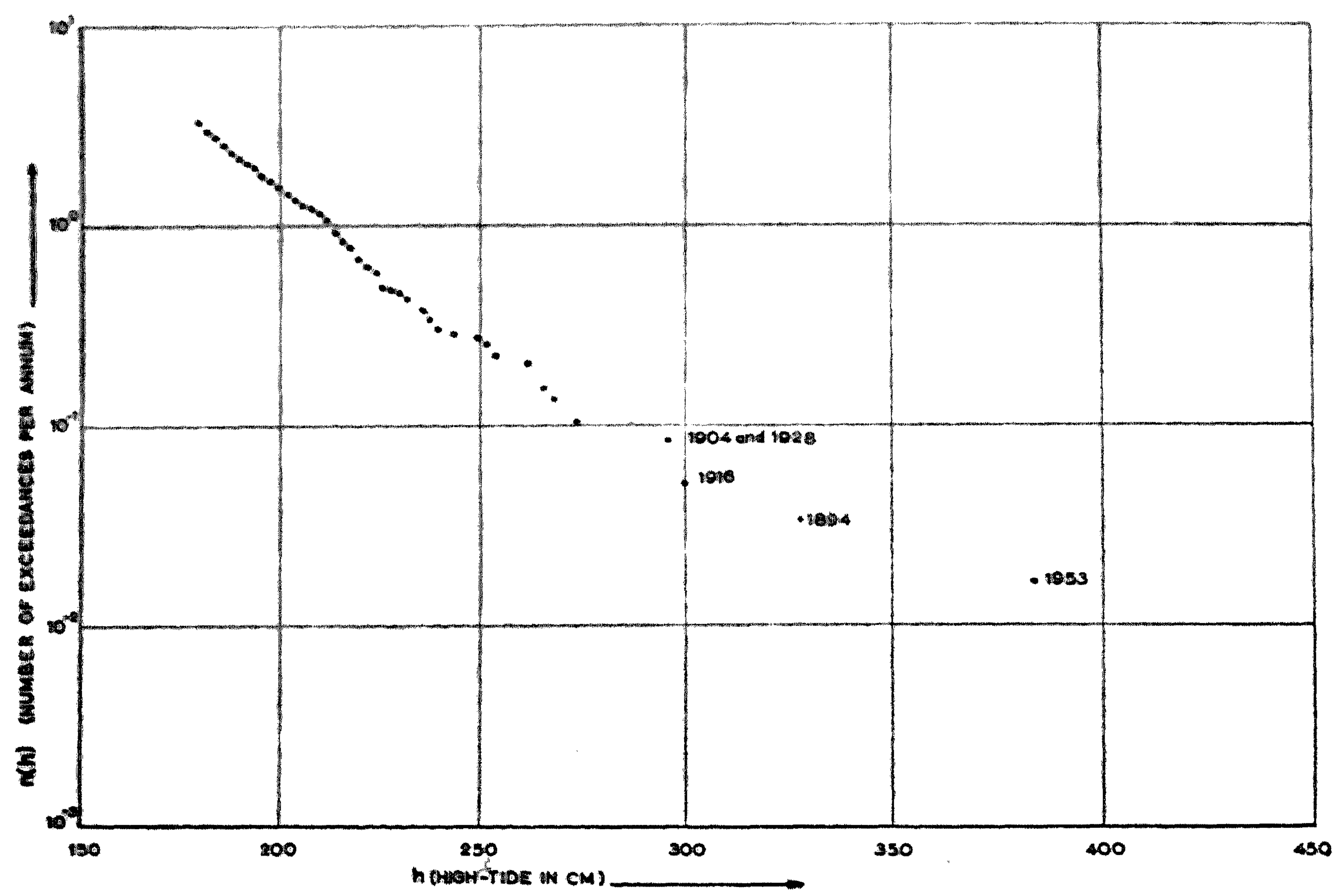

Fig. 1. Exceedance frequencies $\left(\text{plotted at } \frac{i}{60}\right)$ of high tide at Hoek van Holland in the 60 winters of 1888—1939, 1945—1954.

It states (neglecting for the moment the fact that the sea-level at succesive high tides are not independent) that the exceedance probability $1 - F(h)$, where h is the height of the sea-level at high tide above an appropriate zero level, and $F(h)$ its (cumulative) distribution function, is an exponential function $e^{-\alpha h}$. It may also be expressed as a power with any base $g : g^{-h/a_g}$, where $a_g = \alpha^{-1} \ln g$, the difference in height corresponding with probabilities in ratio $1/g$, is a convenient measure, which for $g = 2$, we shortly call the "halving height". At Hoek van Holland it is about 18 centimetres.

When one looks at the graphs one finds that the highest values of h have a tendency to deviate to the right.

One may want therefore to test whether this deviation is significant. The difficulty, caused by the dependence of the order statistics may be overcome in the following way.

Let $\underline{x}_1, \ldots, \underline{x}_n$ [1]) be n independent [2]) continuous "isomorous" variates, i.e. variates all having the same continuous distribution function $F(x)$; let $Q(x) = 1 - F(x)$ be its complement, and $\underline{x}_{(1)}, \ldots, \underline{x}_{(n)}$ the same values arranged according to increasing magnitude:

$$\underline{x}_{(1)} < \underline{x}_{(2)} < \ldots < \underline{x}_{(n)}$$

i.e. the order statistics. For abbreviation we put

$$Q_{(i)} = Q(x_{(i)}).$$

Then, putting for

$$n \geqq i > 1$$

$$H_i = \left(\frac{Q_{(i)}}{Q_{(i-1)}}\right)^{n-i+1}$$

and $H_1 = Q_{(1)}^n$, we have (cf. D. van Dantzig [1947—1950] p. 282) for the element of the simultaneous distribution

$$dF_1 \,.\, dF_2 \,.\, \ldots \,.\, dF_n = dQ_1 \,.\, dQ_2 \,.\, \ldots \,.\, dQ_n = n!\; dQ_{(1)} \,.\, dQ_{(2)} \,.\, \ldots \,.\, dQ_{(n)} = \\ = dH_1 \,.\, dH_2 \,.\, \ldots \,.\, dH_n$$

so that $\underline{H}_1, \underline{H}_2, \ldots, \underline{H}_n$ are independent isomorous variates, all homogeneously distributed over the range (0,1) [3]). In the case where the $\underline{x}_i$ have an exponential distribution, say for $x > 0$:

$$Q(x) = e^{-\alpha x}$$

we have for $i > 1$:

$$H_i = e^{-(n-i+1)\alpha(x_{(i)} - x_{(i-1)})} \text{ and } H_1 = e^{-n\alpha x_{(1)}}$$

so that this implies that the n variates

$$n\underline{x}_{(1)},\; (n-1)(\underline{x}_{(2)} - \underline{x}_{(1)}), \ldots,\; 2(\underline{x}_{(n-1)} - \underline{x}_{(n-2)}),\; \underline{x}_{(n)} - \underline{x}_{(n-1)}$$

are independent and all isomorous with the original $\underline{x}_i$, a special consequence which may more easily be derived directly from the properties of the exponential distribution itself.

As the difference $x_{(n)} - x_{(n-1)}$ between the highest two order statistics in our case is 57 cm, and $2(x_{(n-1)} - x_{(n-2)}) = 56$ cm, these values, with a

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1) The random character of a variable is denoted by underlining its symbol.

2) "Mutually completely independent" according to J. Neyman's terminology.

3) Since this conference was held, I learned from A. Rényi that an equivalent result has also been obtained independently by S. Malmquist [1950] and A. Rényi [1953].

halving height of 18 cm, have exceedance probabilities 0,11 and therefore are by no means out of the common. On the other hand the estimate of the halving height, based on the two differences of the highest values alone (which, of course, because of the small number of data used is very uncertain) would be $\frac{1}{2}(56 + 57) \ln 2 = 39$ cm, i.e. more than double the value found from the lowest values.

According to a remark due to J. Hemelrijk, one has to take account of the fact that this statistical investigation was partly begun because of the flood disaster, i.e. that the series of observations was not terminated at a random moment, but just after a record was reached. This fact might lead to a spurious significance. In order to overcome this difficulty one might either drop the 1953 observation, or otherwise add a number of fictitious observations representing e.g. another ten years. Assuming that then nothing out of the common will happen, these may be drawn at random from the distribution hitherto found. As, however, the highest value does not significantly fall out of the distribution, Hemelrijk's methodologically so important remark can be disregarded in the present case.

Anyhow, the upward tendency of the highest values suggest that the population might be a mixture of two or more different ones, and one may try to unmix these. To this purpose one can first sift out the most dangerous months, which appear to be November, December and January and give rise to a distribution significantly differing from that belonging to the other months. In the second place a selection can be made on meteorological grounds. C. J. van der Ham in the Royal Dutch Meteorological Institute has selected a set of depressions from the period 1888—1939, 1945—1954, which followed paths within a definite geographical strip. His data were analysed in the Mathematical Centre and gave rise to a distribution significantly different from the previous one.

Possibilities of other selections have been investigated in the Mathematical Centre, namely with regard to sun-spot maxima, and to "dangerous years", showing, apart from high maxima, a higher overall average. The first result was completely negative, the latter one, although some indication for its reality was found, could not be established as a significant selection. Like most other statistical results obtained in the Mathematical Centre, this was done in a cooperative study under J. Hemelrijk, to which among others H. Kesten en J. Th. Runnenburg have greatly contributed.

Finally, as A. Benard and Mrs. E. Bos-Levenbach, following a suggestion by J. Hemelrijk, pointed out, the upward tendency of the highest values, which is very often found in similar cases, is, at least partly, a spurious effect, caused by drawing the empirical distribution function, as customary, at the levels $\frac{i}{n+1}\left(\text{or } \frac{i}{n}\right)$. The former choice is usually justified by saying that the distribution function $F(x_{(i)})$ of the i^{th} order statistic $\underline{x}_{(i)}$ has expectation $\frac{i}{n+1}$. As, however, the distributions of the order-statistics are very skew, the median is, for $i > \frac{1}{2}n$ larger and for $i < \frac{1}{2}n$ smaller than the expectation, so that more often than not the empirical distribution will have an S-form rather than be approximately straight. This is a well known phenomenon in most cases where probability paper of any kind is used, and may be overcome by replacing the levels $\frac{i}{n+1}$ by the *medians* instead of the expectations of the $F(x_{(i)})$. These, being the medians of beta-distributions, can easily be computed for any given i and n, and it is found that they differ very little from their asymptotic values, which are

$$\frac{i - \frac{1}{3}}{n + \frac{1}{3}}.$$

Practically it is only important when i or $n - i$ are very small. For $i = 1$, $\frac{\ln 2}{n + 1 - \ln 4} \approx \frac{1 - 0{,}3}{n + 0{,}4}$ is a better approximation.

This replacement of the expectations by the medians may be useful in numerous other cases where some kind of probability paper is used. In our case the deviation to the right at the high values of h does not disappear completely. By the selection mentioned the halving height is raised from 18 to about 23 cm, with a 0,01 upper confidence limit of 28,9 cm. The result is shown in figure 2, again on a logarithmic scale. It is evident that the straight line fits excellently and that the tendency to the right in the highest values is of no importance anymore, so that this must be considered the best estimate of the distribution hitherto available.

One might perhaps object that the emphasis laid here on the deviation of the highest values is somewhat heavy, but it must be remarked firstly, that the guiding principle during the investigation was not connected with these highest values, but with the meteorological background and secondly that the two selected groups (dangerous months, and dangerous types of depressions) both show clearly significant differences with the rest (although, of course, no one would want to suggest that all dangerous cases and only these have been selected). Hence the selections are justified from a purely statistical point of view, even without taking into account *here* the fact that in problems like the present one it is always desirable to remain on the safe side. This fact ought to be (and has been) taken account of in the econometric decision problem.

The question arises whether extrapolation of the distribution now is reliable. From a purely mathematical point of view it is almost trivial that this

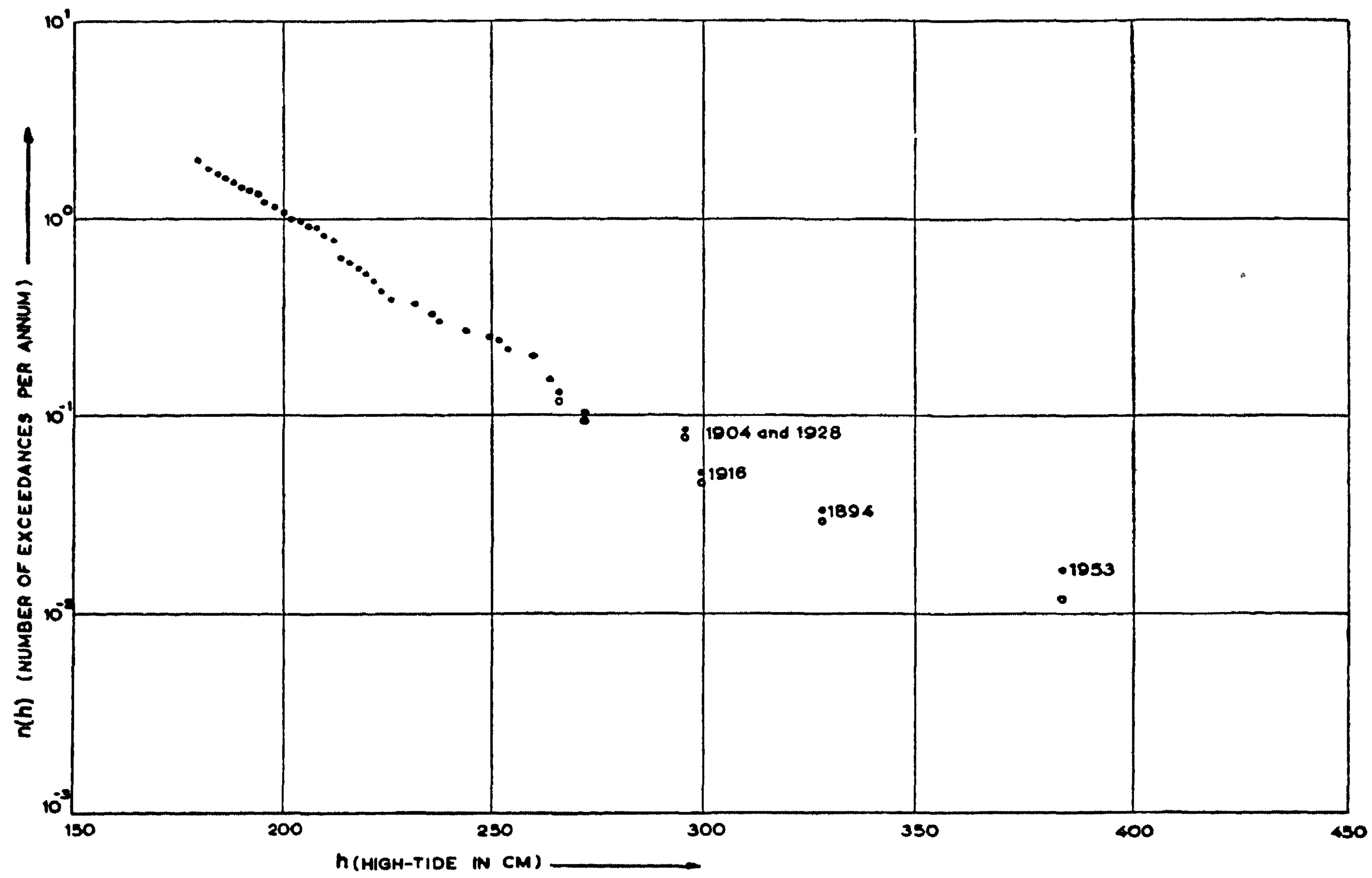

Fig. 2. Exceedance frequencies of high tide during dangerous depressions in the winters of 1888—1939, 1945—1954 at Hoek van Holland. The points are plotted like in fig. 1; the circles denote the plotting positions according to the method of Bos-Benard.

is by no means the case. Even if the most perfect agreement has been found between a finite number of observations and a mathematical distribution function, there is not the slightest guarantee that extrapolation towards the "tails" is allowed. As an example we assume that a perfectly symmetrical die has been thrown n times on a table, and that, instead of the number of eyes, the greatest height above the table is observed. We admitt a small error of measurement, which is normally distributed. If the standard deviation, expressed in the edge of the die as unit is 0,01, the probability that a height $> 1{,}05$ will be found is for all practical purposes negligible. If, however, the edges and angles of the cube are ground off, there is a positive and constant probability that a height $\approx \sqrt{2}$ or even $\approx \sqrt{3}$ will be found. If it has never occurred during the n observations that the die comes to rest on an edge, there is not the slightest statistical evidence for the "bubbles in the tail" of the probability density, which actually are present, and the probability of a height e.g. $> 1{,}4 = 1 + 40\sigma$ is far greater than the best possible estimate. We can only assert for any small positive α with a probability $\geqq 1 - \alpha$ that the probability of the never observed event will be

$$\leqq 1 - \alpha^{\frac{1}{n}} \approx \frac{1}{n} \ln \frac{1}{\alpha}.$$

This perhaps somewhat "pathological" example can be replaced by a more realistic one (cf. D. van Dantzig and J. Hemelrijk [1953]).

Assume that a parameter x on which a windfield depends has a probability density as indicated in fig. 3 below the x-axis, and that the sea level $h = \varphi(x)$

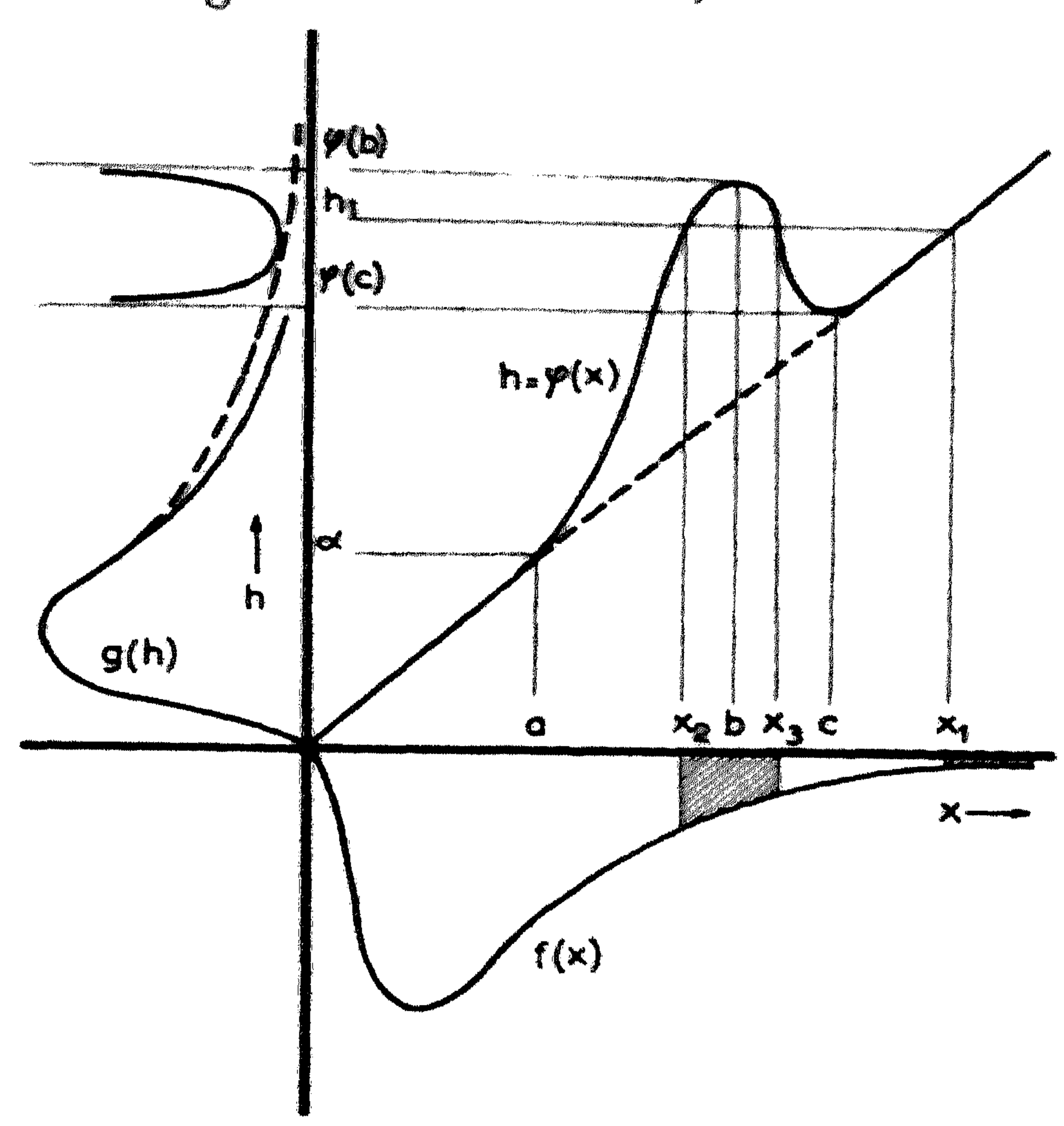

Fig. 3. Influence on a probability distribution of a non-linear transformation.

was found for small values of x ($x \leq a$) hitherto observed to be proportional with x. If this relation would hold for *all* x, the probability distribution of h would (except for a scale factor) be the same as that of x. If, however, $h=\varphi(x)$ actually has a relative maximum at $x = b$ and a relative minimum at $x = c$, the probability density $g(h)$ of h will be different. In fact, it will become infinite at the relative maximal and minimal values of h, i.e. at $h = \varphi(b)$ and $h = \varphi(c)$. The probability that $\underline{h} > h_1$ would then be grossly underestimated by extrapolation of $h = \varphi(x)$ beyond the region of available observations: E.g. $P\{\underline{h} > h_1\}$ would be estimated as $\int_{x_1}^{\infty} f(x)dx$ whereas actually it is $\int_{x_2}^{x_3} f(x)dx$ greater.

Of course in reality the windfield will depend on more than one parameter so that the probability density $g(h)$ of $\underline{h}$ will not really become infinite but only suffer from "tail-bubbling".

A similar form of $\varphi(x)$ will be found in the case where resonance phenomena occur. For his reason it is of the greatest importance to understand the physical causes of the high values of h. In particular we want to know, which types of motion of a depression over the North Sea will be in resonance with the proper oscillations of the basin.

The probability that the height h will not be exceeded during the n independent tides occurring in a year is $(1 - Q(h))^h \approx e^{-nQ(h)} \approx 1 - nQ(h)$ as not only $Q(h)$, but also $nQ(h)$ is small for the larger values of h. By the restriction to independent tides n is not equal to 706, the number of high tides per year, but it is stochastic. We may replace n by its expected value. If only the wintermonths or dangerous depressions are taken, n must accordingly be reduced.

In our case of an exponential distribution the second expression becomes $\exp(-ne^{-\alpha h})$, which has the well-known form of the distribution of extreme values, first derived by R. A. Fisher and L. H. C. Tippett [1928] and extensively studied, generalized and applied to many problems by E. J. Gumbel [1933], [1954].

Fig. 4 shows the distribution of the selected data along the Gumbel-line, which evidently fits well. Determination of the constants by Gumbel's methods, however, gives less accurate results than the direct one. Moreover we see that the last form of the approximation $1 - nQ(h)$ becomes in our case $1 - ne^{-\alpha h}$, i.e. is itself an exponential distribution with the same parameter α (but a different origin), so that we can use this one throughout without going further into the theory of extreme values.

Other distributions than the exponential one, for the high tides have been tried. So J. J. Dronkers [1953] in the Public Works Department has tried a.o. the logarithmically normal distribution, also often used in similar cases. Also the distribution of the difference of the sea level that really occurred and the predicted sea level was investigated by the Public Works Department (cf. F. Volker [1951]). Moreover, the effect of the local wind force has been investigated in the K.N.M.I. by P. J. Rijkoort [1954].

Resuming our statistical results we may say:

The best approximation hitherto obtained to the distributions of high tides and of storm levels is the exponential one, as already found by Wemelsfelder. Although it has no theoretical foundation, and no guarantee at all exists that it will remain valid after considerable extrapolation, there are on the other hand no significant deviations, if the selection of dangerous months and of Van der Ham depressions is used. In particular the **1953** flood does not fall significantly out of the distibution. For these reasons the best thing to do is to use it as long as no significantly deviating results have been found, meanwhile continuing research in this direction, bearing in mind that we have reached here the limits of applicability of statistical methods. For, the relatively

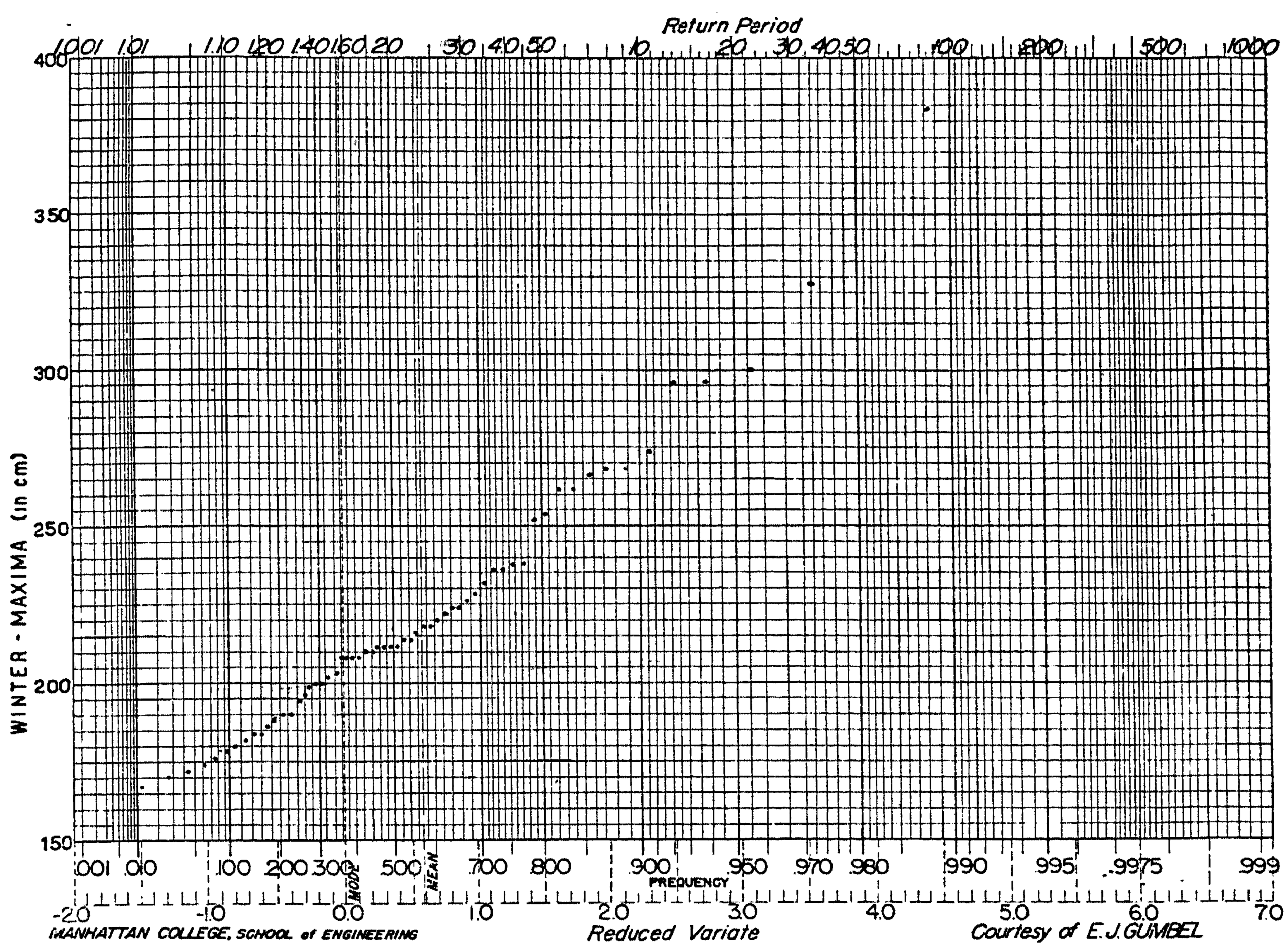

Fig. 4. Winter maxima of high-tide at Hoek van Holland during the period 1888—1939, 1945—1954, plotted according to the Bos-Benard method.

small statistical material which is available makes it impossible to exclude the possibility of slow secular changes in the probability distributions, which nevertheless, if present, might play havoc with all statistical predictions. Apart from known — though insufficiently known — secular phenomena, like the sea level rising and the land level sinking, these can also not be excluded because of the large rôle instabilities play in atmospheric conditions. Thereby e.g. small changes of average temperature in the polar regions might lead to relatively large changes of the probability distribution of the courses and depths of depressions.

It will therefore be necessary to take the physical background into account, in particular to study more closely possible resonance and interference phenomena, which are known to exist. Whereas statistical methods suffer from the drawback that they can be applied to *past experience* only, physical methods allow us to find out, which phenomena could possibly occur in future under any imaginable realistic circumstances, and under which circumstances these might become dangerous.

So this is the point where statistics must pass the torch to hydrodynamics.

The hydrodynamic problem.

The hydrodynamic problem is: to determine the elevation of the sea-level if a depression of given form and intensity moves over the North Sea, and, in particular, to find those motions of depressions which give rise to especially high elevations, i.e. to obtain a better insight in the physical background of storm surges on the North Sea. This problem had already been studied before the 1953 flood by W. F. Schalkwijk [1947] in the Meteorological Institute. His paper has become fundamental to all later research. He considered more in particular the stationary solutions of the linearized equations for a wind-field, constant in space and time.

The problem can be attacked only by means of considerable simplifications. In the first place the non-linear hydrodynamical equations are replaced by their linear approximation. Although some non-linear effects are known, e.g. the fact that the influence of wind is greater at low tide than at high tide, they are also known to be rather small. Because of this linearization we may consider the difference only between the actual elevation and the one due to the astronomical tide, i.e. we may leave the astronomical tide out of consideration. In the second place the vertical component of the velocity of the water is neglected, and the horizontal components are replaced by their average values over a vertical column reaching from the bottom to the level of the sea. Thirdly internal friction and friction caused by the coasts are neglected, and only friction due to the bottom is taken account of. Moreover the irregular form of the North Sea is replaced by a rectangular basin, attached to an ocean represented by a half plane. This approximation is quite reasonable, if the "leak" at the Dover Straits is neglected.

The non-linear equations governing the problem were recently discussed anew in the Public Works Department by J. C. Schönfeld [1954]. We shall here, however, consider the twodimensional linearized equations only.

Then, be ζ the elevation of the sea level above a given zero level, x and y the rectangular coordinates on the surface of the sea, u and v the average components of the velocity, h the depth of the sea, gX and gY the components of the external force per unit of mass, consisting of the gradient of the atmospheric pressure, and the shearing force exerted by the windfield. The latter is *not* proportional to the wind velocity, but approximately to its square. We shall, however, not go into this relationship, but assume X and Y to be *given* functions of x, y, t. The equations of continuity and of motion have the form

$$
\begin{aligned}
&\frac{\partial(hu)}{\partial x} + \frac{\partial(hv)}{\partial y} + \frac{\partial\zeta}{\partial t} = 0 \\
&\frac{\partial u}{\partial t} + \lambda u - \Omega v + g\frac{\partial\zeta}{\partial x} = gX \qquad (1)\\
&\frac{\partial v}{\partial t} + \lambda v + \Omega u + g\frac{\partial\zeta}{\partial y} = gY.
\end{aligned}
$$

Here g is the acceleration of gravity, λ a friction coefficient, roughly proportional with h^{-1}, and Ω the coefficient of Coriolis, viz. $\Omega = 2\omega \sin\varphi$ where ω is the angular velocity of the rotation of the earth and φ the latitude. Although Ω and h, hence also λ depend on x and y, we shall consider the case only where they are constant (small basin of constant depth). Because of the linearity of the equations the tidal motion may be omitted.

The North Sea is considered (fig. 5) as the rectangle

$$R: \quad |x| \leqq a, \quad 0 < y < b$$

where $x = -a$, $y = 0$, $x = +a$ roughly represent the English, the Dutch and the Scandinavian coasts respectively, whereas $y = b$ is the open end of the sea at the Atlantic Ocean. Roughly $2a \approx 450$ km, $b \approx 4a \approx 900$ km. The "leak" at the Dover Straits is at present left out of consideration.

The boundary conditions state that the normal component of the velocity vanishes along the coasts, and that ζ is continuous along the ocean frontier. If the ocean is considered as infinitely deep, we may assume $\zeta = 0$ there.

Hence

$$
(2) \qquad \Gamma \begin{cases} \Gamma_1 : & |x| < a,\ y = b \qquad \zeta = 0 \\ \Gamma_2 \begin{cases} x = \pm a,\ 0 < y < b & u = 0 \\ |x| < a,\ y = 0 & v = 0. \end{cases} \end{cases}
$$

We shall not introduce dimensionless variables, as sometimes one type of normalization, e.g. $\Omega = gh = 1$, sometimes another one, e.g. $a = \frac{1}{2}\pi$, is the more useful one.

We consider the stationary case first. The equations reduce to

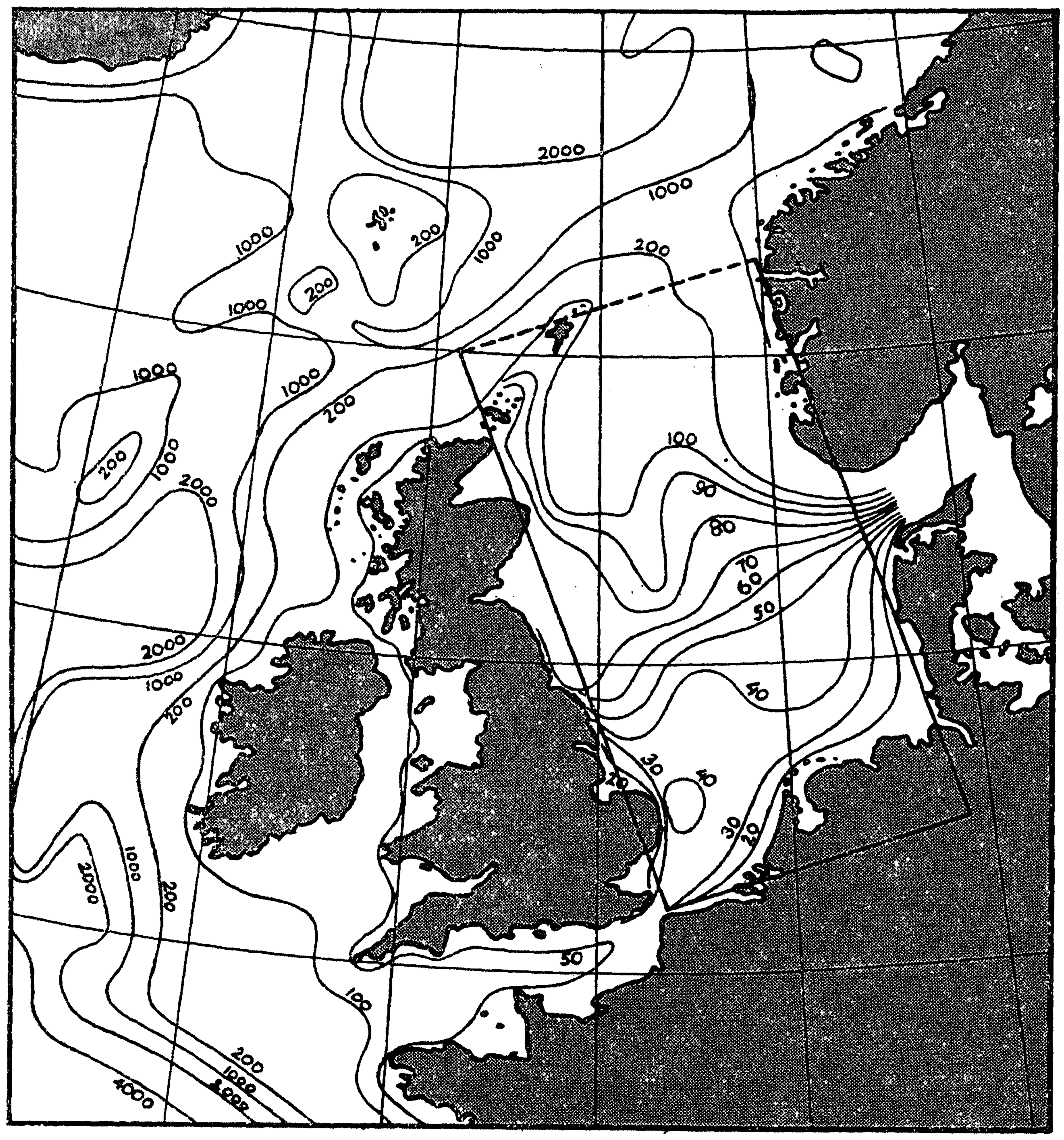

Fig. 5. The North Sea with approximating rectangle and depth in metres.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

(3) $$\lambda u - \Omega v + g\frac{\partial \zeta}{\partial x} = gX$$

$$\lambda v + \Omega u + g\frac{\partial \zeta}{\partial y} = gY.$$

Whereas Schalkwijk had already studied the case of constant X and Y, M. P. H. Weenink [1954], working in the Meteorological Institute with P. Groen, recently studied the case where $X = 0$ throughout, $Y = \text{const} = 2W$ on the Western half of the North Sea, and $Y = 0$ on the Eastern half, supposing the sea 1^0 to be closed, or 2^0 to border an ocean, represented by a half plane of a depth equal to that of the sea. In both cases he found ζ in the origin (which is the point most important for our country) to have the *same* value as in the case of a constant windfield, having the same overall average value (hence $Y = W$ everywhere on the North Sea). The conformal mapping of a half plane with adjoining rectangle needed in this case had already been studied in the Mathematical Centre by G. W. Veltkamp [1953a].

The general theory is being developed in the Applied Mathematics Department of the Mathematical Centre under my direction. Firstly the case of stationary windfields on a sea of constant or sectionally constant depth was studied by G. W. Veltkamp [1953b, 1954]. As a two dimensional vectorfield (X, Y) can always be described by means of two potentials U and V, determining its divergence-free and curl-free parts respectively:

$$X = -\frac{\partial U}{\partial y} - \frac{\partial V}{\partial x},\ Y = \frac{\partial U}{\partial x} - \frac{\partial V}{\partial y},$$

the solution can be obtained in the case of a closed sea from a complex analytic function L of $z = x + iy$ as

$$\zeta = \operatorname{Re}\frac{\Omega}{\lambda} + i\,(L + U + iV), \tag{4}$$

where L must satisfy the boundary condition

$$U + \operatorname{Re} L = \text{const.} \tag{5}$$

along each closed part of the boundary. The computations could be carried out for some special examples of circular windfiels above an ocean, represented by a halfplane. As an example fig. 6 shows the lines of constant ζ for the case $V = 0,\ U = -\frac{1}{2}b^3\,(|z - ia\,|^2 + b^2)^{-1}$, representing a divergence-free windfield with a non-singular centre. In the case of a sea with sectionally constant depth, Veltkamp found that L must be replaced by a sectionally holomorphic function, and that the Plemelj formulae induce a singular integral equation along the line of discontinuity of the depth. If a shallow bay (the North Sea) borders an ocean (the Atlantic) of finite but large depth, an iteration procedure using the ratio of the two depths (both assumed to be constant) can be applied.

If in the case of a rectangular bay as mentioned before (where one side is about twice as long as the other, which implies that the Jacobian elliptic function which maps the rectangle on a half-plane nearly degenerates) the potentials U and V are chosen in such a way that $U = 0$ on Γ (which is always possible) then the function $L(z)$ proves to be nearly a constant in the vicinity of "the South Coast", viz.

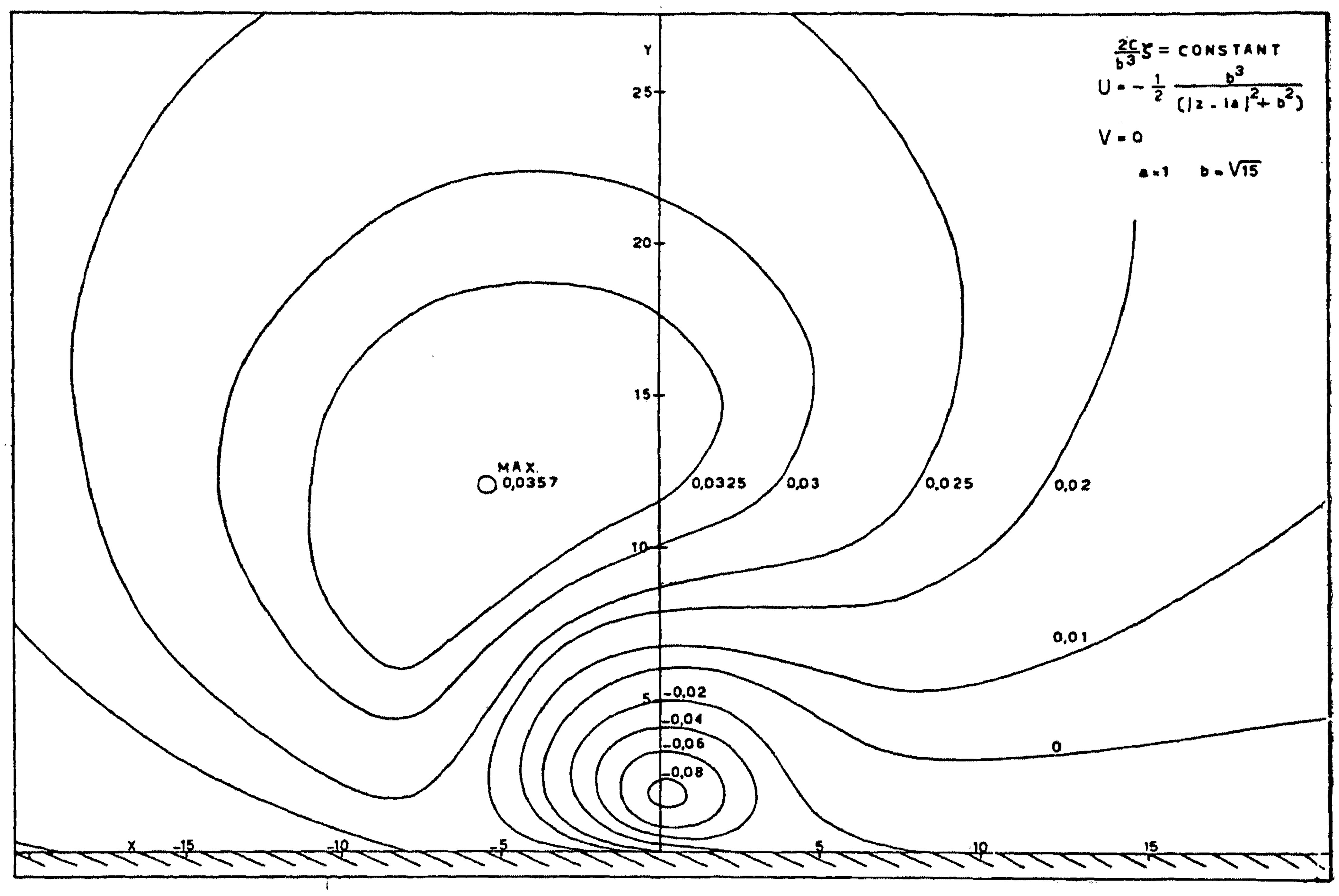

Fig. 6. A stationary windfield on a half plane. Lines of constant ζ.

$$L \sim -\tfrac{1}{2}i \cos \pi\gamma \int_0^a \left[\left(\operatorname{tg}\frac{\pi t}{4a}\right)^{-2\gamma} V(t-a,\, b) - \left(\operatorname{tg}\frac{\pi t}{4a}\right)^{2\gamma} V(a-t,\, b)\right] dt$$

with
$$\gamma = \frac{1}{\pi} \operatorname{arc\,tg} \frac{\Omega}{\lambda}, \quad 0 < \gamma < \tfrac{1}{2}.$$

Accordingly ζ can be computed in the point $x = y = 0$. For a few types of windfields the numerical work was carried out in the Computation Department of the Mathematical Centre.

For the case of a linear windfield

$$X = \tfrac{1}{2}W\left(1 - \frac{x}{2a} - \frac{2y-b}{b}\right)$$
$$Y = -W\left(1 - \frac{5}{4}\frac{x}{a}\right),$$

with $b = 4a$, sketched in fig. 7, the result was

$$\zeta = 1{,}20\,\frac{bW}{h},$$

whereas the homogeneous windfield

$$X = \tfrac{1}{2}W \quad Y = -W$$

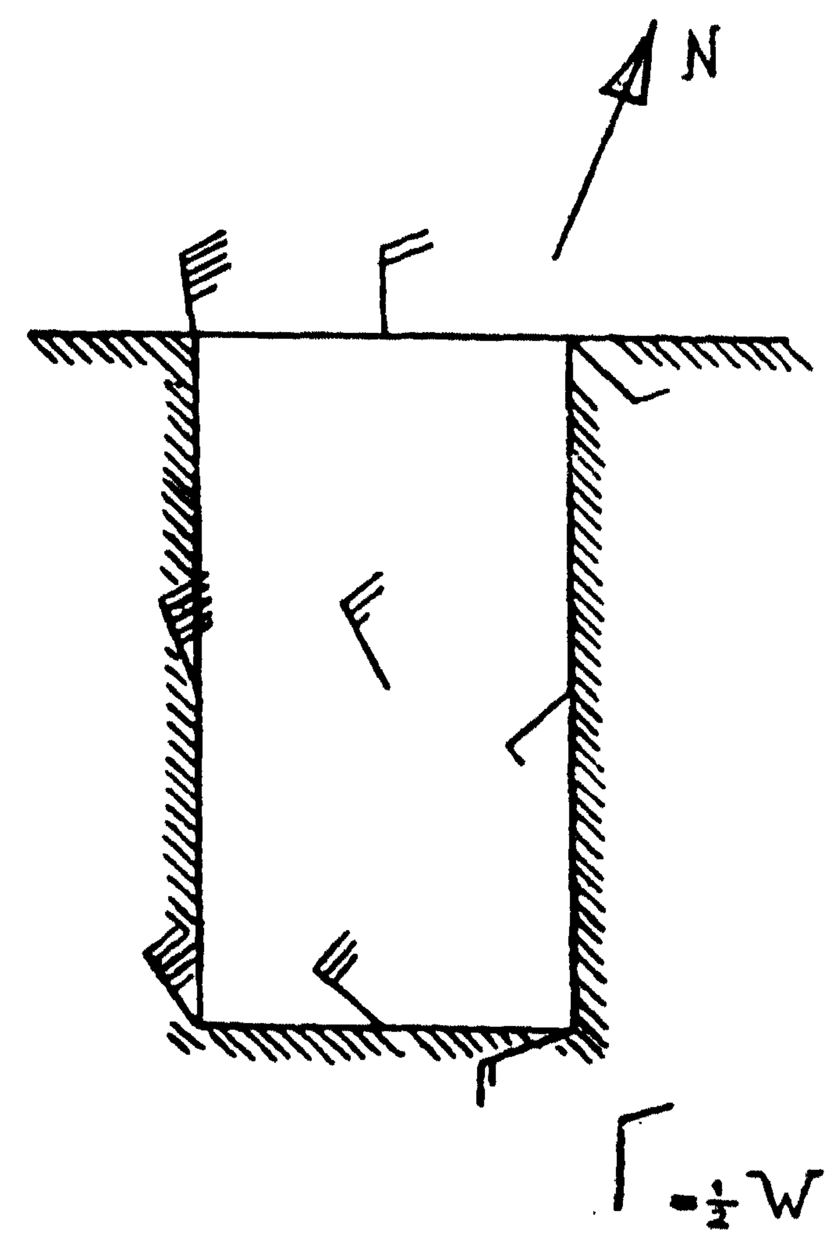

Fig. 7. Linear windfield on rectangular sea.

would have given $\zeta = 1{,}105\,\frac{bW}{h}$,

and the field

$$X = 0 \quad Y = -W$$

led to $\zeta = 1 \cdot \frac{bW}{h}$. Hence we see that the cross component $x = \frac{1}{2}W$ adds a $10\frac{1}{2}\%$ to the elevation and the non-homogeneity another $9\frac{1}{2}\%$.

I now come to the non-stationary case which was attacked a few months ago in the Mathematical Centre.

By means of a Laplace transformation

$$\bar{\zeta}(x, y, p) = \int_0^\infty e^{-pt}\zeta(x, y, t)dt$$

(6) $$\zeta(x, y, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\bar{\zeta}(x, y, p)dp,$$

and similarly for u and v, we obtain the equations

$$\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} + \frac{p}{h}\bar{\zeta} = 0$$

(7) $$(p + \lambda)\bar{u} - \Omega\bar{v} + g\frac{\partial\bar{\zeta}}{\partial x} = g\bar{X}$$

$$(p + \lambda)\bar{v} + \Omega\bar{u} + g\frac{\partial\bar{\zeta}}{\partial y} = g\bar{Y}.$$

In the first place it is of importance to determine the proper oscillations of the basin by putting $X = Y = 0$, $p = -\lambda + i\omega$. Normalizing the dimensions by putting

$$gh = 1, \; a = \tfrac{1}{2}\pi$$

we find that $\bar{u}$, $\bar{v}$ and $\bar{\zeta}$ all satisfy the differential equation

$$\Delta\bar{u} - k^2\bar{u} = 0 \tag{8}$$

with

$$k^2 = \frac{1}{gh}\left(1 + \frac{i\lambda}{\omega}\right)(\Omega^2 - \omega^2) \quad (\mathrm{Re}\, k > 0). \tag{9}$$

Without friction and without Coriolis force we would simply have $k = \dfrac{i\omega}{\sqrt{gh}}$. Coriolis changes the relation into $k^2 = \dfrac{\Omega^2 - \omega^2}{gh}$, and the friction is the cause of the occurrance of the denominator ω. In other terms: because of the friction the differential equation in x, y, t is no longer of the second order, but of the third order with respect to time. If the variation of depth had also been taken into account first order derivatives with respect to x and y would have occurred too, and the equations would have been of fifth order with respect to time.

The proper oscillations can be found by generalizing slightly a method introduced by G. I. Taylor [1922] in 1922 already for an infinitely long channel, closed at one end, without friction ($\lambda = 0$).

Putting formally

$$\bar{u} = \sum_{-\infty}^{\infty}{}_n \; i^n e^{inx}\{u_n^+ e^{y\sqrt{k^2+n^2}} + u_n^- e^{-y\sqrt{k^2+n^2}}\} \tag{10}$$

($\mathrm{Re}\sqrt{k^2 + n^2} > 0$), the differential equation $(\Delta - k^2)\bar{u} = 0$ is satisfied, as well as the boundary conditions $\bar{u} = 0$ if $x = \pm\tfrac{1}{2}\pi$, provided u_n^+ and u_n^- are odd functions of n. The analogous quantities $v_n^{\pm}$ can easily be expressed in the $u_n^{\pm}$ by means of factors

$$\frac{n\Omega + \omega\sqrt{k^2 + n^2}}{n\omega + \Omega\sqrt{k^2 + n^2}},$$

and in the two "Kelvin-waves", i.e. the solutions of (7) (for $X = Y = 0$), together with $\bar{u} = 0$.

By means of the identities for $|z| < \frac{1}{2}\pi$

$$e^{(2m-1)iz} = \frac{1}{\pi} \sum_{-\infty}^{\infty}{}_n \frac{(-1)^{m-n-1}}{m-n-\frac{1}{2}} e^{2inz},$$

the two other boundary conditions can be introduced, the $u_n^{\pm}$ are found to vanish for all even n, and the other values have to be found from an infinite sequence of equations. The characteristic values of ω, i.e. the poles of the Laplace transform have to be computed as the roots of an infinite determinant (D. van Dantzig [1954*a*]). The computation has not yet been carried out.

We now return to the non-homogeneous equations (7). By elimination of $\bar{u}$ and $\bar{v}$ we obtain the differential equation

$$\Delta\bar{\zeta} - k^2\bar{\zeta} = \bar{F} \tag{11}$$

$$k^2 = \frac{p}{gh}\left(p + \lambda + \frac{\Omega^2}{p+\lambda}\right)$$

$$F = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} - \frac{\Omega}{p+\lambda}\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right), \tag{12}$$

whereas the boundary conditions become: $\bar{\zeta} = 0$ along the ocean Γ_1 and

$$\frac{\partial\bar{\zeta}}{\partial\nu} + \frac{\Omega}{p+\lambda}\frac{\partial\bar{\zeta}}{\partial s} = \bar{f} \tag{13}$$

with

$$f = \begin{cases} X + \dfrac{\Omega}{p+\lambda} Y \text{ for } x = \pm a,\ 0 < y < b \\[2ex] \dfrac{\Omega}{p+\lambda} X - Y \text{ for } |x| < a,\ y = 0. \end{cases} \tag{14}$$

The main difficulties are caused by the presence of the force of Coriolis, and consist of the occurrence of the skew boundary conditions (13) instead of the usual normal ones. If the force of Coriolis were negligible we should have $\Omega = 0$, and the problems were relatively simple, but it is not, and sometimes it is even the greatest term determining the motion.

Application of Green's theorem with an as yet undetermined function $G = G(x, y, \xi, \eta)$ having a logarithmic singularity at $x = \xi$, $y = \eta$, use of the boundary conditions (13) and partial integration gives

$$\bar{\zeta} = \varphi - \iint_R \bar{\zeta}(\Delta - k^2)G\, d\xi d\eta + \\ + \int_{\Gamma_2} \bar{\zeta}\left(\frac{\partial G}{\partial\nu} - \frac{\Omega}{p+\lambda}\frac{\partial G}{\partial s}\right) ds - \int_{\Gamma_1} G\frac{\partial\bar{\zeta}}{\partial\nu}\, ds, \tag{15}$$

$$\varphi = \iint_R G\bar{F}\, d\xi d\eta - \int_{\Gamma_2} G\bar{f} ds.$$

If we could find a Green's function G, satisfying

$$(16)\qquad \begin{aligned} \Delta G - k^2 G &= 0 \text{ in } R \\ G &= 0 \text{ on } \Gamma_1 \\ \frac{\partial G}{\partial \nu} - \frac{\Omega}{p+\lambda}\frac{\partial G}{\partial s} &= 0 \text{ on } \Gamma_2, \end{aligned}$$

(15) would yield the solution explicitly. We did, however, not succeed in obtaining this G explicitly. So we have to be content with less.

The following methods offer themselves.

1. Taking

$$G = \frac{1}{2\pi}\{K_0(kr) - K_0(kr')\},$$

where K_0 is Bessel's function of the third kind, zero order and imaginary argument, whereas

$$r^2 = (x-\xi)^2 + (y-\eta)^2,\ r'^2 = (x-\xi)^2 + (y+\eta-2b)^2,$$

the differential equation and the boundary condition along the ocean are satisfied. Thereby (15) becomes a Cauchy-singular integral equation along the coast, which may be solved by the methods Poincaré [1910] introduced for similar problems in the theory of tides. These methods have been studied extensively by the Tiflis school under I. N. Mushkhelishvili [1953]. Notably results obtained by I. N. Vekua [1939] and B. V. Khvedelidze [1943] are important for our purpose.

2. A solution of the differential equations (16), satisfying the boundary conditions along the two long sides $x = \pm a$ of the rectangle may be found; it yields a singular integral equation along the two other sides.

3. The same can be done for the two other sides of the rectangle. As Veltkamp pointed out, this is better, as the main effects in the more important cases will be a current in the North-South direction.

4. Finally we can satisfy all boundary conditions by a harmonic function G. Then (15) becomes

$$\bar{\zeta} = \bar{\varphi} + k^2 \iint \bar{\zeta}\, G\, d\xi d\eta,$$

i.e. a *non-singular* integral equation, which for small k^2 may be solved explicitly by the method of iterated kernels. The solutions G in the three last-mentioned cases have all been obtained in the Mathematical Centre by H. A. Lauwerier [1954]. In the last case we have

$$G = \text{Re}\,\frac{1}{4\pi}\int_{M(z)}^{\infty}\left\{\frac{A}{s - M(\zeta)} - \frac{A^*}{s - M(\zeta)^*}\right\} s^{-\frac{\omega}{\pi}}\, ds$$

where $z = x + iy$, $\zeta = \xi + i\eta$ and $M(z)$ a double-periodic function, the periods of which are twice the sides of the rectangle. The asterisk denotes the complex conjugate. For the North Sea, where roughly $b \approx 4a$ the elliptic function $M(z)$ very nearly degenerates into trigonometric functions.

The solution for small k^2, i.e. small p, which we obtain in this way may be transformed back into an asymptotic expansion of ζ, valid for large t. This, however, is of relatively little importance, as we are not particularly interested in the height the sea-level reaches after the storm is over. In combination, however, with the determination of the proper oscillations the path of the inverse Laplace transformation can be drawn over one or more of the corresponding points $p = -\lambda + i\omega$ and ther by will lead to solutions in which the main effect is incorporated.

Lauwerier also clarified the main cause of the difficulty of the problem, due to the skew boundary condition. Taking first the case of a half plane $x > 0$ with a skew boundary condition in a constant (real) direction, Green's function has not only a logarithmic singularity in the image $(-\xi, \eta)$ or (ξ, η), but it makes also a finite jump along a half line ending in this point, which can be considered as being covered by dipoles.

If we pass to a quarter plane we get apart from the logarithmic singularities obtained by reflection also an algebraic singularity at the corner.

This, Mr Chairman, shows how far we had come just before the Congress began.

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